PERTH COLLEGE

Year 12

Semester One Examination 2011

Question/Answer booklet

MATHEMATICS 3CMAT/3DMAT

Section One (Calculator - free)

Student Name: SOLUTIONS (FINAL)

Time allowed for this section

Reading time before commencing work: 5 minutes
Working time for paper: 50 minutes

Material required/recommended for this section

To be provided by the supervisor
Question/answer booklet for Section One
Formula sheet which may also be used for Section Two

To be provided by the candidate
Standard items: *pens, pencils, pencil sharpener, highlighter, eraser, ruler*

Special items: *nil*

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further

Structure of this paper

	Number of questions available	Number of questions to be attempted	Suggested working time (minutes)	Marks available
Section One Calculator-free	**8**	**8**	**50 minutes**	**40**
Section Two Calculator-assumed	X	X	100 minutes	80
			Total marks	120

Instructions to candidates

1. Answer the questions in the spaces provided.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. They can be used for planning your responses and/or as additional space if required to continue an answer
   a. Planning: If you use the spare pages for planning, indicate this clearly at the top of the page.
   b. Continuing an answer: If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answers you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams

Question 1 **[1, 3 = 4 marks]**

Consider the following functions:

$$g(x) = (x-1)^2 + 1 \qquad \text{and} \qquad h(x) = \sqrt{x-5}$$

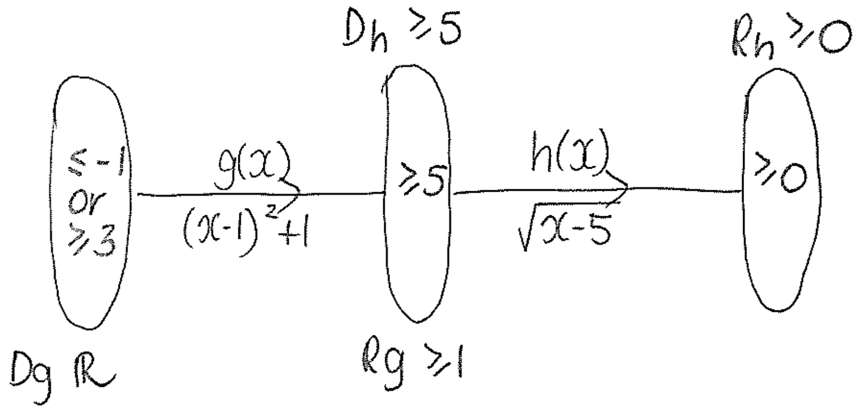
a) Determine the exact value of $hg(4)$.

$g(4) = (4-1)^2 + 1 = 10$

$h(10) = \sqrt{10-5} = \sqrt{5}$

$\therefore hg(4) = \sqrt{5}$

b) State the domain and range of $hg(x)$.



$(x-1)^2 + 1 = 5$

$(x-1)^2 = 4$

$x - 1 = \pm 2$

$x = \pm 2 + 1$

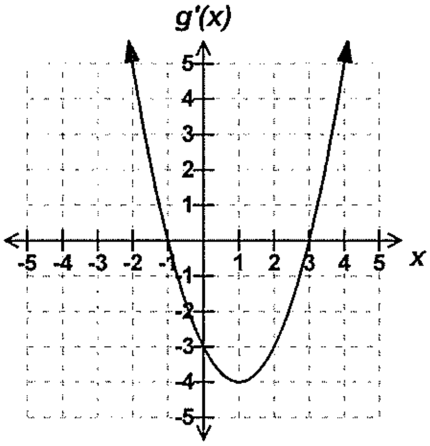
$x = -1, 3.$

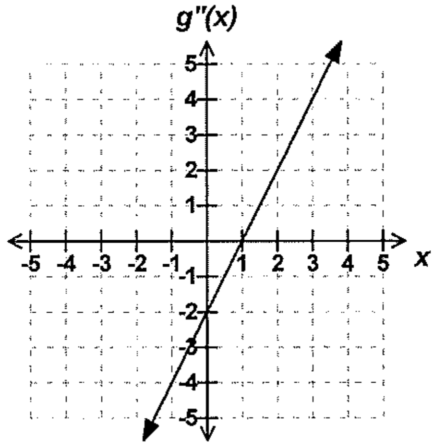
$D = \{x : x \leq -1 \text{ OR } x \geq 3, x \in \mathbb{R}\}.$

$R = \{y : y \geq 0, y \in \mathbb{R}\}.$

Question 2 **[1, 1, 3 = 5 marks]**

Let $y = g(x)$ be defined as some function of x.
The graph of g has an inflection point at **P**, a local minimum at **M** and a local maximum at **Q**.
Sketches of the graphs of $g'(x)$ and $g''(x)$ are shown below.





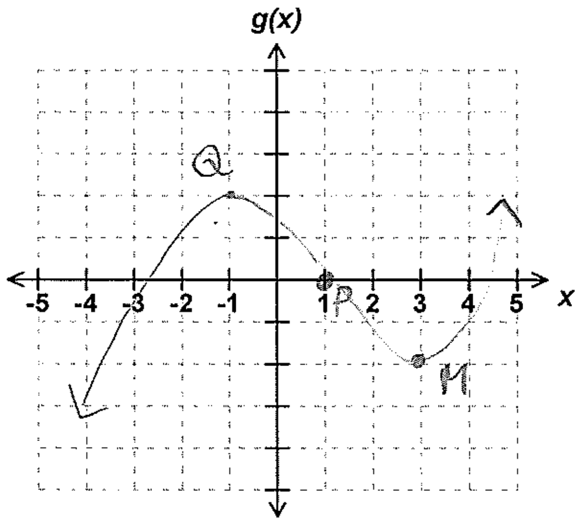
a) Use the information above to determine:

(i) the ***x*-coordinate** of **P**. $x = 1$.

(ii) the ***x*-coordinate** of **M**. $x = 3$.

b) Given that $g(1) = 0$, sketch a possible graph of $g(x)$. On your sketch, show the points **P, M** and **Q**, labeling each one clearly.

Q ⇒ max @ (-1, y)
P ⇒ PI @ (1,0)
M ⇒ min. @ (3, y)
shape



Question 3 **[9 marks]**

a) Differentiate $\dfrac{2p-5}{p^2+3p-1}$ with respect to p. Fully simplify the numerator of your answer.

[3]

$$\frac{2(p^2+3p-1)-(2p+3)(2p-5)}{(p^2+3p-1)^2}$$

$$=\frac{2p^2+6p-2-4p^2+10p-6p+15}{(p^2+3p-1)^2}$$

$$=\frac{-2p^2+10p+13}{(p^2+3p-1)^2}$$

b) Determine the values of c and d given that $\dfrac{d}{dx}(4x\sqrt{2x+3}) = \dfrac{cx+d}{\sqrt{2x+3}}$.

[4]

$$4\sqrt{2x+3}+\frac{4x}{\sqrt{2x+3}}$$

$$=\frac{4(2x+3)+4x}{\sqrt{2x+3}}$$

$$=\frac{12x+12}{\sqrt{2x+3}} \qquad \therefore c=12 \ \& \ d=12$$

c) Find $\dfrac{dy}{dx}$ given $y=(3e^{4x}+2)^5$. Fully simplify your answer.

[2]

$$\frac{dy}{dx}=5(3e^{4x}+2)^4 \cdot 12e^{4x}$$

$$=60e^{4x}(3e^{4x}+2)^4$$

Question 4 **[2, 4 = 6 marks]**

The points (-2, 1), (1, -2) and (3, 16) all lie on the parabola $f(x) = ax^2 + bx + c$.

a) Use this information to form three equations in terms of *a*, *b* and *c*.

$1 = a(-2)^2 + b(-2) + c \quad \therefore 1 = 4a - 2b + c. \quad ①$

$-2 = a(1)^2 + b(1) + c \quad \therefore -2 = a + b + c. \quad ②$

$16 = a(3)^2 + b(3) + c \quad \therefore 16 = 9a + 3b + c. \quad ③$

(-1 / ERROR)

b) Solve these equations to determine the values of *a*, *b* and *c*.

$① - ② \Rightarrow 3 = 3a - 3b. \quad ④$

$① - ③ \Rightarrow -15 = -5a - 5b \quad ⑤$

$④ \times 5 \Rightarrow 15 = 15a - 15b.$

$⑤ \times 3 \Rightarrow -45 = -15a - 15b.$ +.

$-30 = -30b$

$\therefore b = 1$

sub $b = 1$ into $3a - 3b = 3$

$3a - 3 = 3$

$3a = 6$

$a = 2,$

sub $b = 1$ & $a = 2$ into

$4a - 2b + c = 1.$

$8 - 2 + c = 1.$

$c = -5.$

$\therefore a = 2, b = 1, c = -5.$

Question 5 **[1, 2, 2 = 5 marks]**

Determine each of the following indefinite integrals. Express your answers with positive indices.

a) $\int (x^3 - 2x - 4)\, dx$ $= \frac{x^4}{4} - x^2 - 4x + c$

b) $\int 3x^2 (4x^3 - 5)^7\, dx$ $= \frac{(4x^3-5)^8}{8} \times \frac{1}{4}$

$= \frac{(4x^3-5)^8}{32} + c$

c) $\int \frac{2x^3 - 4x}{\sqrt{x}}\, dx$ $= \int 2x^{5/2} - 4x^{1/2}\, dx$

$= 2x^{7/2} \times \frac{2}{7} - 4x^{3/2} \times \frac{2}{3}$

$= \frac{4}{7} x^{7/2} - \frac{8}{3} x^{3/2} + c$

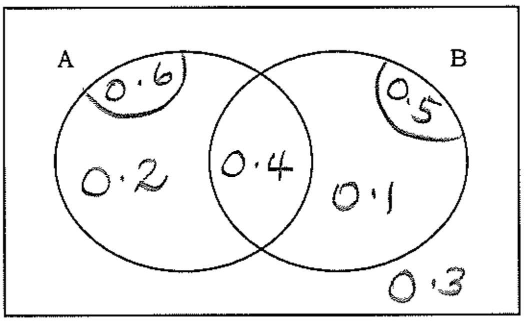
-1 overall if +c omitted (even once)

Question 6 **[5 marks]**

a) For the events A and B represented in the Venn diagram below,

P(A $\cap$ B) = 0.4, P(A) = 0.6 and P(A | B) = 0.8.

1



Determine

(i) P(B) $P(A|B) = \dfrac{P(A \cap B)}{P(B)}$ [2]

$\therefore P(B) = \dfrac{0.4}{0.8}$ $P(B) = 0.5$

(ii) P($\overline{A \cup B}$) $= 0.3$ [1]

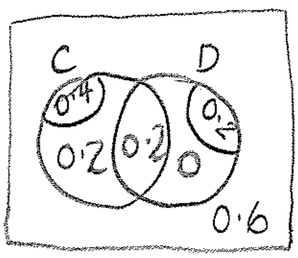
b) Consider another two events C and D such that P ($\overline{C} \cap \overline{D}$) = 0.6 , P (C) = 0.4 and P (D) = 0.2. Determine whether or not C and D are mutually exclusive events and justify your answer mathematically.

[2]

1

No, not mutually exclusive

$P(A \cap B) = 0.2$



Question 7 **[2 marks]**

The point $(3,\ 2e^2)$ lies on the graph of $f(x) = 2e^{x-1}$.

If the following transformations are applied to $f(x)$ **in succession**, what would be the co-ordinates of the resulting location of this point?

- Reflection about the *x*-axis
- Horizontal translation of 5 units right
- Vertical translation of 4 units down
- Reflection about the *y*-axis

x: $+5$ $x-1$.

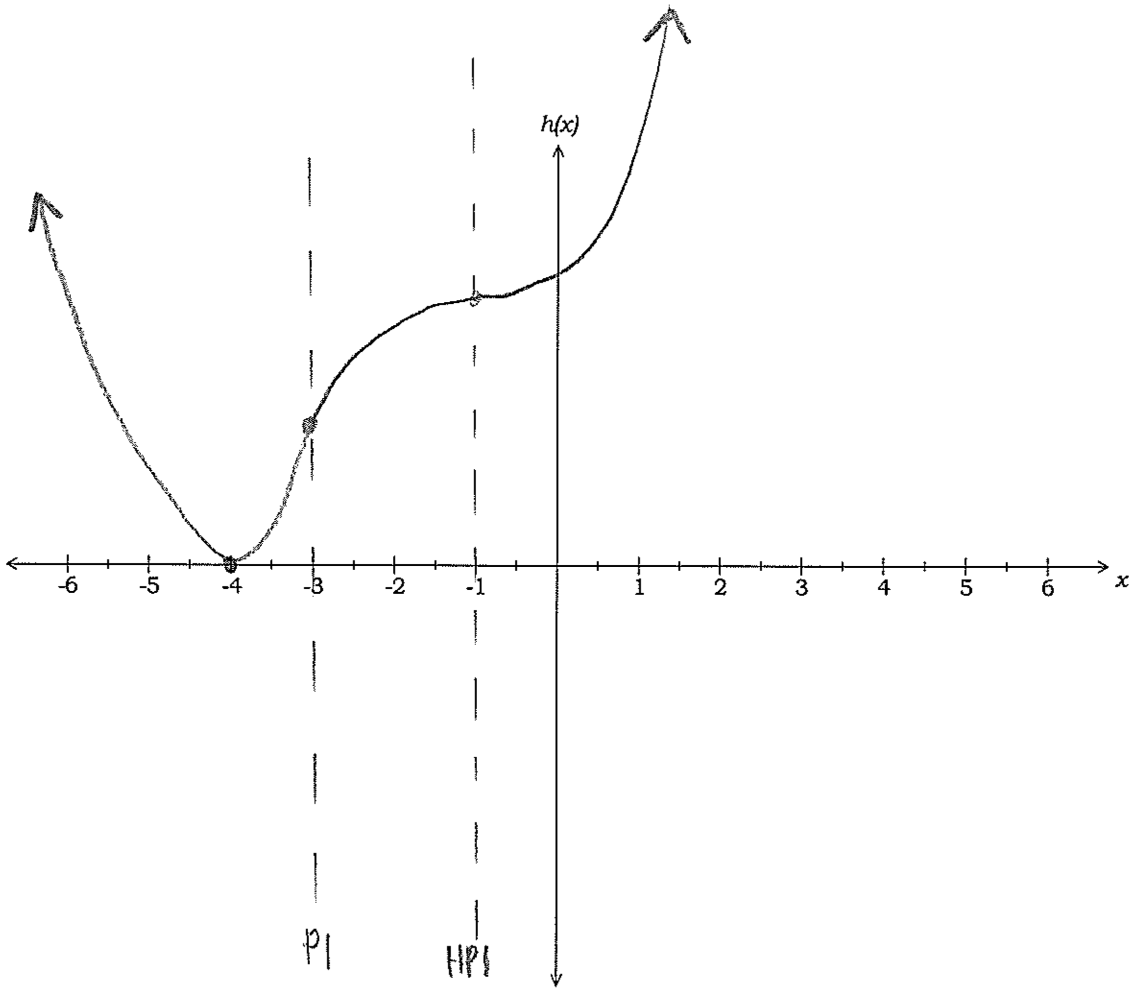
y: $x-1$. -4.

$(-8,\ -2e^2-4)$

Question 8 **[4 marks]**

Draw a neat sketch of a function $y = h(x)$ which satisfies **all** of the following conditions:

- $h(-4) = h'(-4) = 0$
- $h'(-1) = h''(-1) = 0$
- $h''(-3) = 0$
- $h'(x) < 0$ <u>only</u> for $x < -4$



END OF SECTION ONE

PERTH COLLEGE

Year 12

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Question/Answer booklet

MATHEMATICS 3CMAT/3DMAT

Section Two (Calculator - assumed)

Student Name: SOLUTIONS (FINAL)

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Special items: *drawing instruments, templates, notes (two unfolded sheets of A4 paper) and up to three calculators (CAS, graphic or scientific) which satisfy the conditions set by the Curriculum Council for this course.*

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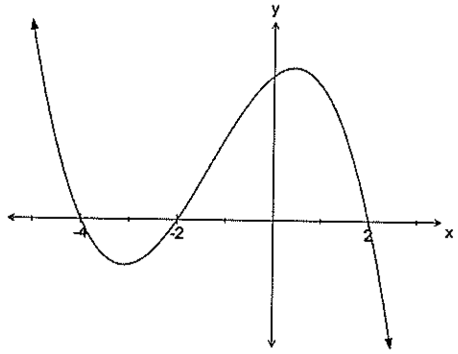
Question 9 **[1, 1, 1, 3 = 6 marks]**

The graph of $y = f(x)$ is shown below.



Given $\int_{-4}^{-2} f(x)\ dx = -2$ and $\int_{-2}^{2} f(x)\ dx = 5$, determine each of the following:

a) $\int_{-4}^{2} f(x)\ dx$ $= -2 + 5$

$= 3.$

b) $\int_{-2}^{2} k f(x)\ dx$, where k is some constant

$= 5k$

c) $\int_{-4}^{2} |(f(x)|\ dx$ $= 2 + 5$

$= 7$

d) $\int_{2}^{-2} x - f(x)\ dx$ $= \int_{2}^{-2} x\, dx - \int_{2}^{-2} f(x)\, dx$

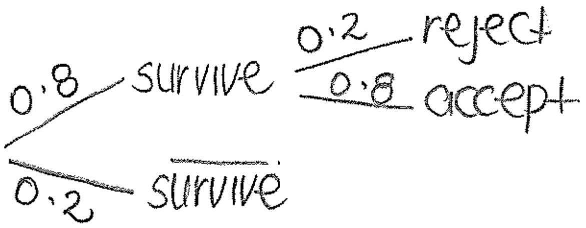
$= \left[\frac{x^2}{2}\right]_{2}^{-2} - (-5)$

$= \frac{(-2)^2}{2} - \frac{2^2}{2} - (-5)$

$= 5$

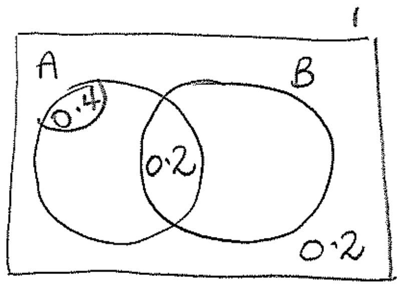
Question 10 **[6 marks]**

a) The probability of surviving a particular organ transplant operation is 0.8. If a patient survives the operation, the probability that his or her body will reject the transplanted organ within the first month is 0.2. What is the probability that a particular patient survives the operation and their body does not reject the organ? [2]



$$P(\text{survives and accepts}) = 0{\cdot}8 \times 0{\cdot}8$$
$$= 0{\cdot}64.$$

b) Given that $P(A \cup B) = 0.8$, $P(A \cap B) = 0.2$ and $P(B|A) = 0.5$, determine whether or not the events A and B are independent. Justify your answer mathematically. [4]



$$P(B|A) = \frac{P(B \cap A)}{P(A)}$$

$$\therefore 0{\cdot}5 = \frac{0{\cdot}2}{P(A)}$$

$$P(A) = \frac{0{\cdot}2}{0{\cdot}5}$$

$$\therefore P(A) = 0{\cdot}4$$

$$P(B) = 0{\cdot}6.$$

$$P(A) \times P(B) = 0{\cdot}4 \times 0{\cdot}6$$
$$= 0{\cdot}24$$
$$\neq P(A \cap B)$$
$$(\text{OR } P(B|A) \neq P(B))$$

$\therefore$ Not independent

Question 11 **[5 marks]**

The size of a population of bacteria that is introduced to a nutrient grows according to the formula $P(t) = 5000 + \dfrac{3000\,t}{100 + t^2}$, where t is the time, measured in hours, after the introduction.

a) Find the rate at which the population is changing 2 hours into the experiment. Show all working.

[3]

$$\frac{dp}{dt} = \frac{3000(100+t^2) - 2t(3000t)}{(100+t^2)^2}$$

$$\left.\frac{dp}{dt}\right|_{t=2} = \frac{3000(1+2^2) - 2(2)(3000(2))}{(100+2^2)^2}$$

$= 26 \cdot 63$ /h (2dp)

(Answer only 1m)

b) Find the average rate at which the population is changing over the first four hours.

[2]

$$\frac{P(4) - P(0)}{4-0}$$

$$= \frac{5103 \cdot 448 - 5000}{4}$$

$\approx 25 \cdot 86$ /h (2dp)

Question 12 **[9 marks]**

Consider the function $h(x) = x^4 - 2x^3 + 1$ defined over the interval $-1.5 \le x \le 2.5$.

a) Use Calculus techniques to determine the co-ordinates and nature of any stationary points.

[5]

$h'(x) = 4x^3 - 6x^2$

solve $4x^3 - 6x^2 = 0$

$x = 0$ OR $x = 1.5$

$h(0) = 1 \quad h(1.5) = -0.6875$

$h''(x) = 12x^2 - 12x$

$h''(0) = 0 \quad h''(1.5) = 9$ (OR sign test)

$\therefore (0, 1)$ is a HPI

$(1.5, -0.6875)$ is a min

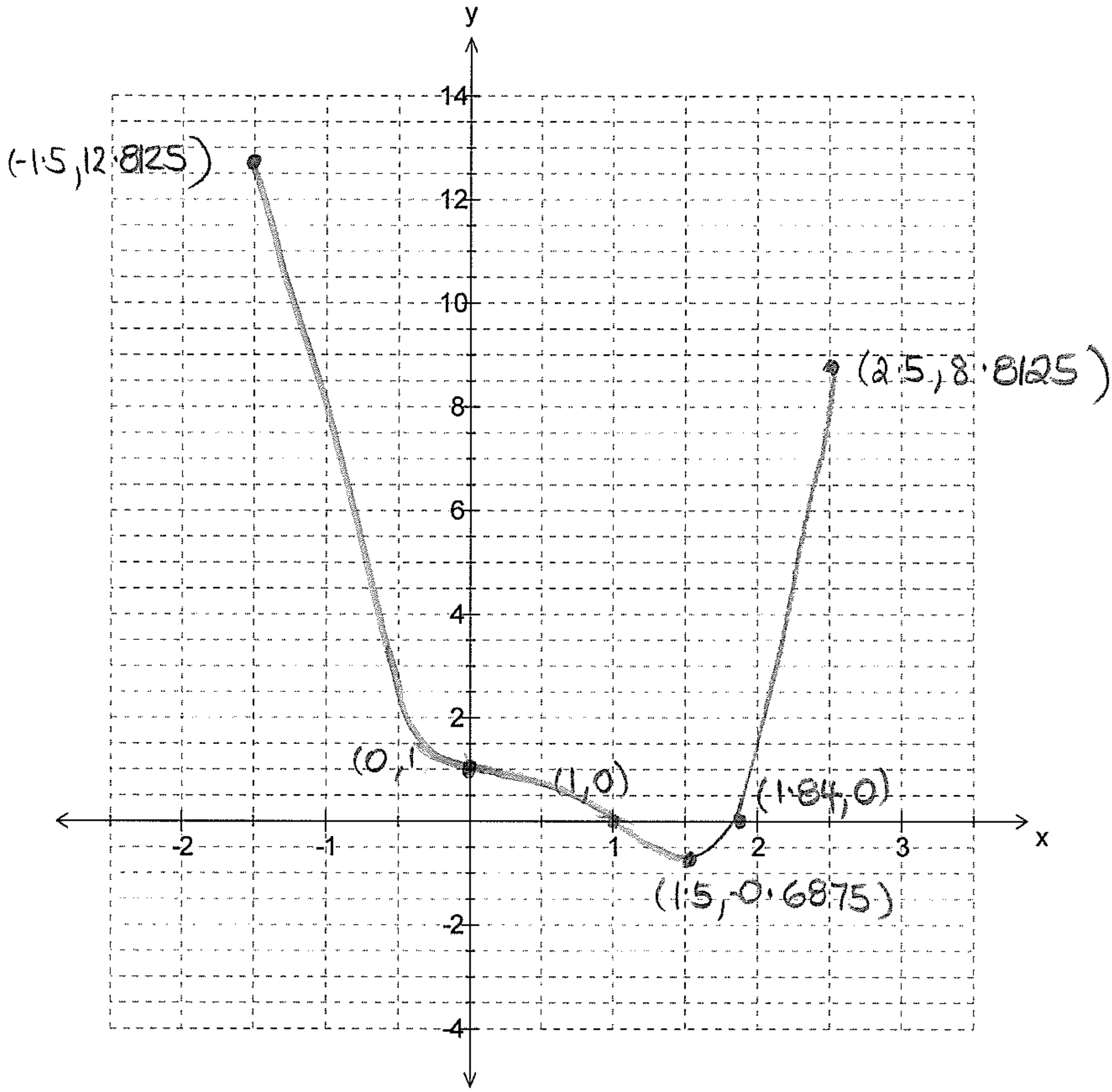
b) Determine the global maximum of $h(x)$.

[1]

12.8125 (when $x = -1.5$)

c) Draw a neat sketch of $h(x)$, clearly indicating all stationary points, points of inflection, intercepts and endpoints.

[3]



Question 13 **[9 marks]**

Subsets are formed by choosing letters from the word F A C E T I O U S (which is a very interesting word because it contains all five vowels in alphabetical order!)

a) How many **four** letter subsets can be formed? [1]

$\binom{9}{4} = 126$

b) How many **four** letter subsets can be formed which contain at least 1 vowel? [2]

$\binom{9}{4} - \underbrace{\binom{5}{0}\binom{4}{4}}_{\text{no vowels}} = 125$

c) How many **five** letter subsets can be chosen from the word F A C E T I O U S which are also subsets of the word F A C T O R I S E ? [1]

$\binom{8}{5} = 56.$

The letters within the subsets can now be arranged to form different "words". From the letters in the word
F A C E T I O U S :

d) How many different five letter "words" can be formed? [1]

$\underline{9}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{5} = 15\,120.$

One of these five letter words is chosen at random. What is the probability that it contains:

e) both the letters C and E, but with the C and E separated by exactly one letter? [2]

$$\frac{\underset{\substack{C\\E}}{2}\ \ 7\ \ \underset{\substack{C\\E}}{1}\ \ 6\ \ 5\ \times 3}{\binom{2}{2}\binom{7}{3}5!} = \frac{1260}{4200}$$

f) both the letters C and E, but with the C and E separated by at least one letter? [2]

$1 - P(C \,\&\, E \text{ are adjacent})$

$$= 1 - \frac{\underset{\substack{C\\E}}{2}\ \ \underset{\substack{C\\E}}{1}\ \ 7\ \ 6\ \ 5\ \times 4}{4200}$$

$$= 1 - \frac{1680}{4200}$$

$$= \frac{2520}{4200}$$

Question 14 **[5 marks]**

a) In a certain medical procedure, a tracer dye is injected into the pancreas to measure its function rate. The amount of dye remaining in the pancreas, D grams, at any time t minutes after the injection has been administered is given by the equation
$D = D_0 e^{kt}$.
In a pancreas that is functioning normally, 4% of the dye will be excreted each minute.

If a dosage of 0.5g of dye is administered to a patient, how much dye will be secreted after one hour if the patient's pancreas is functioning normally?
Answer correct to 3 dp.

[3]

$D = 0.5e^{-0.04t}$

when $t = 60$ $D = 0.045g$

$\therefore$ amount secreted $= 0.5 - 0.045$

$= 0.455g$ (3dp)

b) The rate at which the concentration , C units, of a drug in the blood is reduced by normal metabolism is proportional to the value of this concentration at any time t measured in hours. That is, $\frac{dC}{dt} = kC$.
Given that the concentration drops from 2 400 units to 2 000 units in the first hour, express C as a function of t, including the value of any constants. Express answers correct to 3dp where necessary.

[2]

$C = C_0 e^{kt}$

$\therefore C = 2400e^{kt}$

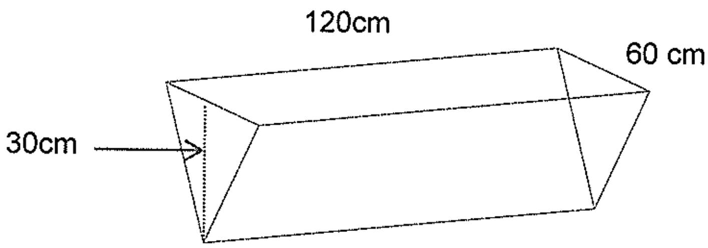
when $t = 1$, $C = 2000$

$\therefore 2000 = 2400e^{k}$ $\therefore k = -0.182$ (3dp)

$\therefore C = 2400e^{-0.182t}$

Question 15 [2, 4 = 6 marks]

A water trough 60 cm across, 120 cm long, and 30 cm deep has ends in the shape of isosceles triangles. (See the diagram below.)



Let the depth of the water in the trough be d.

a) Clearly show that the volume of water in the trough at any depth d is given by the equation

$$V = 120d^2$$

base of Δ = $2 \times$ height of Δ (similar Δs)

$V = \frac{1}{2} \times 2d \times d \times 120 = 120d^2$

b) If the depth of water in the trough is reduced by 5%, determine the percentage change in the volume of the water using the Incremental Formula.

$\Delta d = -0.05d$.

$\frac{dV}{dd} = 240d$.

$\Delta V \approx \Delta d \times \frac{dV}{dd}$.

$\approx -0.05d \times 240d$.

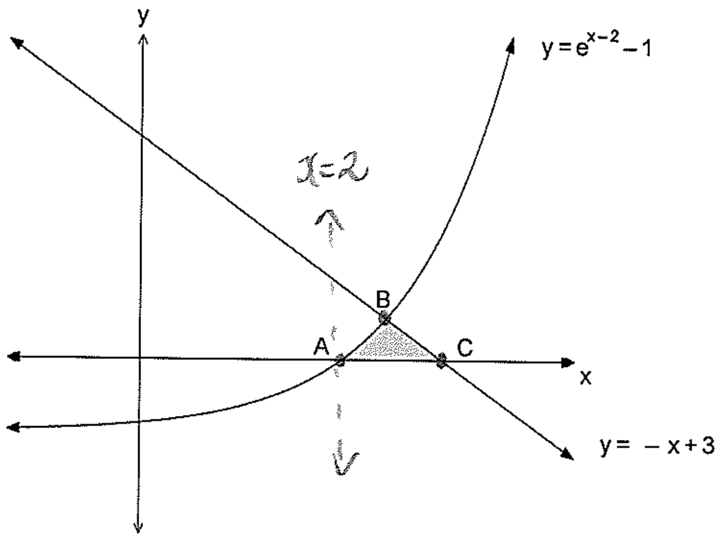
$\approx -12d^2$

$\therefore \% \Delta V \approx \frac{-12d^2}{120d^2} \times 100$.

$\approx -10\%$. (OR 10% ↓)

Question 16 **[3, 3, 2, 2 = 10 marks]**

Consider the diagram below, which shows the graphs of $y = e^{x-2} - 1$ and $y = -x + 3$.



a) Determine the co-ordinates of A, B (to 2 dp) and C.

A(2,0) B(2·44, 0·56) C(3,0)

b) Write an expression, involving integrals, to determine the area of the shaded region. Use your expression to determine the area.

$$\text{Area} = \int_{2}^{2\cdot44} (e^{x-2} - 1)\,dx + \int_{2\cdot44}^{3} (-x+3)\,dx$$

$= 0\cdot270$ units^2 (3dp)

c) Determine the area enclosed by $y = e^{x-2} - 1$, $y = -x + 3$ and the line $x = 2$. Clearly show how you used integrals to obtain your answer.

[2]

$$\text{Area} = \int_{2}^{2.44} (-x+3) - (e^{x-2} - 1)\, dx$$

$$= 0.230 \text{ units}^2 \text{ (3dp)}$$

d) The function $y = e^{x-2} - 1$ undergoes a series of transformations and the resulting curve has the equation $y = 2e^{x+1} - 2$. Clearly describe the transformations that have occurred, in correct order, to obtain this new function.

[2]

Horizontal translation 3 ←

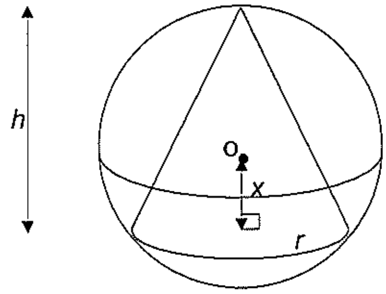
($x-2 \underline{+3}$ becomes $x+1$)

vertical dilation factor 2

($2(e^{f(x)} - 1)$ becomes $2e^{f(x)} - 2$)

Question 17 **[7 marks]**

A cone just fits inside a sphere of radius 10 cm.
Let the height of the cone be h cm and the radius of the cone be r cm.
O is the centre of the sphere.



a) Show that the expression for the volume of the *cone* can be given by
$V = \frac{\pi}{3}(100 - x^2)(10 + x)$.

[2]

$r^2 = 10^2 - x^2 = 100 - x^2$

$h = 10 + x$

$V = \frac{1}{3}\pi r^2 h = \frac{\pi}{3}(100 - x^2)(10 + x)$ as required

b) Find the exact dimensions of the cone which has maximum volume. Justify your answer using Calculus.

[5]

$\frac{dV}{dx} = \frac{\pi}{3}\left[-2x(10 + x) + 1(100 - x^2)\right]$

solve $\frac{dV}{dx} = 0$

$x = \cancel{-10}$ OR $x = \frac{10}{3}$

$\frac{dV}{dx} = (-20x - 3x^2 + 100)\frac{\pi}{3}$

$\therefore \frac{d^2V}{dx^2} = (-20 - 6x)\frac{\pi}{3}$ $\quad \frac{d^2V}{dx^2}\Big|_{x=\frac{10}{3}} = -\frac{40\pi}{3}$ $\therefore$ max

$\therefore$ radius $= \sqrt{100 - \left(\frac{10}{3}\right)^2} = \sqrt{\frac{800}{9}}$ cm (simplify surd not necessary)

height $= 10 + \frac{10}{3} = \frac{40}{3}$ cm

Question 18 **[3, 3 = 6 marks]**

Given that $y = x^2 e^{2x}$,

a) show that the derivative can be expressed in the form $\frac{dy}{dx} = axe^{2x}(1 + cx)$ and determine the values of a and c.

$$\frac{dy}{dx} = 2xe^{2x} + 2e^{2x}.x^2$$
$$= 2xe^{2x}(1 + x)$$

$\therefore c = 1.$ & $a = 2$

b) Determine the equation of the tangent to the curve $y = x^2 e^{2x}$ at the point $(1, e^2)$.

$$\left.\frac{dy}{dx}\right|_{x=1} = 2e^2(2)$$
$$= 4e^2$$

Eqn of tangent is $y = 4e^2 x + c$.

sub in $(1, e^2)$

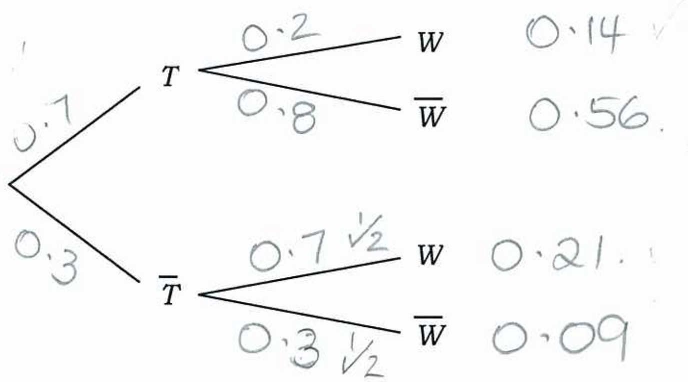
$$e^2 = 4e^2 + c$$
$$\therefore c = -3e^2$$

$\therefore$ Eqn of tangent is $y = 4e^2 x - 3e^2$

Question 19 [3, 1, 2 = 6 **marks]**

Tom is 5 years old and often throws a tantrum to get what he wants. The probability that Tom throws a tantrum over any encounter with his mum is 0.7. Given that he throws a tantrum, the probability he gets his own way is 0.2. Irrespective of whether or not he throws a tantrum, Tom gets his own way 35% of the time.

Let T be the event "throws a tantrum" and W be the event "gets his own way".



a) Draw a well labelled tree diagram, showing all branch and end of branch values to represent the above information.

b) Determine the probability that Tom throws a tantrum or gets his own way.

$P(T \cup W) = 1 - 0.09$

$= 0.91$

(OR $0.14 + 0.56 + 0.21$)

c) $P(T \mid \overline{W}) = \dfrac{0.56}{0.65}$

Question 20 **[5 marks]**

Given that $\int_{2.5}^{k} e^{2x-5}\ dx = \frac{e-1}{2}$, find the value of k.

Clearly show working to support your answer.

$$\int_{2.5}^{k} e^{2x-5}dx = \left[\frac{e^{2x-5}}{2}\right]_{2.5}^{k}$$

$$\therefore \frac{e-1}{2} = \frac{e^{2k-5}}{2} - \frac{e^{2(2.5)-5}}{2}$$

$$\therefore \frac{e-1}{2} = \left(\frac{e^{2k-5}}{2}\right) - \left(\frac{e^{0}}{2}\right)$$

$$\therefore \frac{e-1}{2} = \frac{e^{2k-5}-1}{2}$$

$\therefore 2k-5 = 1$

$2k = 6$

$\therefore k = 3$